

Maximal Green Sequences of Exceptional Finite Mutation Type Quivers^{*}

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Abstract. Maximal green sequences are particular sequences of mutations of quivers which were introduced by Keller in the context of quantum dilogarithm identities and independently by Cecotti–Córdova–Vafa in the context of supersymmetric gauge theory. The existence of maximal green sequences for exceptional finite mutation type quivers has been shown by Alim–Cecotti–Córdova–Espahbodi–Rastogi–Vafa except for the quiver X_7 . In this paper we show that the quiver X_7 does not have any maximal green sequences. We also generalize the idea of the proof to give sufficient conditions for the non-existence of maximal green sequences for an arbitrary quiver.

Key words: skew-symmetrizable matrices; maximal green sequences; mutation classes

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1 Introduction and main results

Maximal green sequences are particular sequences of mutations of quivers. They were used in [9] to study the refined Donaldson–Thomas invariants and quantum dilogarithm identities. Moreover, the same sequences appeared in theoretical physics where they yield the complete spectrum of a BPS particle, see [5, Section 4.2]. The existence of maximal green sequences for exceptional finite mutation type quivers has been shown in [1] except for the quiver X_7 . In this paper, we show that the quiver X_7 does not have any maximal green sequences. We also give some general sufficient conditions for the non-existence of maximal green sequences for an arbitrary quiver.

To be more specific, we need some terminology. Formally, a quiver is a pair $Q = (Q_0, Q_1)$ where Q_0 is a finite set of vertices and Q_1 is a set of arrows between them. It is represented as a directed graph with the set of vertices Q_0 and a directed edge for each arrow. We consider quivers with no loops or 2-cycles and represent a quiver Q with vertices $1, \dots, n$, by the uniquely associated skew-symmetric matrix $B = B^Q$ defined as follows: the entry $B_{i,j} > 0$ if and only if there are $B_{i,j}$ many arrows from j to i ; if i and j are not connected to each other by an edge then $B_{i,j} = 0$. We will also consider more general skew-symmetrizable matrices: recall that an $n \times n$ integer matrix B is skew-symmetrizable if there is a diagonal matrix D with positive diagonal entries such that DB is skew-symmetric. To define the notion of a green sequence, we consider pairs (\mathbf{c}, B) , where B is a skew-symmetrizable integer matrix and $\mathbf{c} = (\mathbf{c}_1, \dots, \mathbf{c}_n)$ such that each $\mathbf{c}_i = (c_1, \dots, c_n) \in \mathbb{Z}^n$ is non-zero. Motivated by the structural theory of cluster algebras, we call such a pair (\mathbf{c}, B) a Y -seed. Then, for $k = 1, \dots, n$ and any Y -seed (\mathbf{c}, B) such that all entries of \mathbf{c}_k are non-negative or all are non-positive, the Y -seed mutation μ_k transforms (\mathbf{c}, B) into the Y -seed $\mu_k(\mathbf{c}, B) = (\mathbf{c}', B')$ defined as follows [8, equation (5.9)], where we use the notation $[b]_+ = \max(b, 0)$:

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- the entries of the exchange matrix $B' = (B'_{ij})$ are given by

$$B'_{ij} = \begin{cases} -B_{ij} & \text{if } i = k \text{ or } j = k, \\ B_{ij} + [B_{ik}]_+ + [B_{kj}]_+ - [-B_{ik}]_+ - [-B_{kj}]_+ & \text{otherwise;} \end{cases} \quad (1)$$

- the tuple $\mathbf{c}' = (\mathbf{c}'_1, \dots, \mathbf{c}'_n)$ is given by

$$\mathbf{c}'_i = \begin{cases} -\mathbf{c}_i & \text{if } i = k, \\ \mathbf{c}_i + [\text{sgn}(\mathbf{c}_k)B_{k,i}]_+ \mathbf{c}_k & \text{if } i \neq k. \end{cases} \quad (2)$$

The matrix B' is skew-symmetrizable with the same choice of D . We also use the notation $B' = \mu_k(B)$ (in (1)) and call the transformation $B \mapsto B'$ the *matrix mutation*. This operation is involutive, so it defines a *mutation-equivalence* relation on skew-symmetrizable matrices.

We use the Y -seeds in association with the vertices of a regular tree. To be more precise, let \mathbb{T}_n be an n -regular tree whose edges are labeled by the numbers $1, \dots, n$, so that the n edges emanating from each vertex receive different labels. We write $t \overset{k}{-} t'$ to indicate that vertices $t, t' \in \mathbb{T}_n$ are joined by an edge labeled by k . Let us fix an initial seed at a vertex t_0 in \mathbb{T}_n and assign the (initial) Y -seed (\mathbf{c}_0, B_0) , where \mathbf{c}_0 is the tuple of standard basis. This defines a Y -seed pattern on \mathbb{T}_n , i.e. an assignment of a Y -seed (\mathbf{c}_t, B_t) to every vertex $t \in \mathbb{T}_n$, such that the seeds assigned to the endpoints of any edge $t \overset{k}{-} t'$ are obtained from each other by the seed mutation μ_k ; we call (\mathbf{c}_t, B_t) a Y -seed with respect to the initial Y -seed (\mathbf{c}_0, B_0) . We write:

$$\mathbf{c}_t = \mathbf{c} = (\mathbf{c}_1, \dots, \mathbf{c}_n), \quad B_t = B = (B_{ij}).$$

We refer to B as the *exchange matrix* and \mathbf{c} as the *\mathbf{c} -vector* tuple of the Y -seed. These vectors have the following *sign coherence property* [7]:

$$\text{each vector } \mathbf{c}_j \text{ has either all entries nonnegative or all entries nonpositive.} \quad (3)$$

Note that this property is conjectural if B is a general non-skew-symmetric (but skew-symmetrizable) matrix. It implies, in particular, that the Y -seed mutation in (2) is defined for any Y -seed (\mathbf{c}_t, B_t) , furthermore \mathbf{c}_t is a basis of \mathbb{Z}^n [10, Proposition 1.3]. We also write $\mathbf{c}_j > 0$ (resp. $\mathbf{c}_j < 0$) if all entries are non-negative (resp. non-positive).

Now we can recall the notion of a green sequence [3]:

Definition 1. Let B_0 be a skew-symmetrizable $n \times n$ matrix. A *green sequence* for B_0 is a sequence $\mathbf{i} = (i_1, \dots, i_l)$ such that, for any $1 \leq k \leq l$ with $(\mathbf{c}, B) = \mu_{i_{k-1}} \circ \dots \circ \mu_{i_1}(\mathbf{c}_0, B_0)$, we have $\mathbf{c}_{i_k} > 0$, i.e. each coordinate of \mathbf{c}_{i_k} is greater than or equal to 0; here if $k = 1$, then we take $(\mathbf{c}, B) = (\mathbf{c}_0, B_0)$. A *green sequence for a quiver* is a green sequence for the associated skew-symmetric matrix.

A green sequence $\mathbf{i} = (i_1, \dots, i_l)$ is maximal if, for $(\mathbf{c}, B) = \mu_{i_l} \circ \dots \circ \mu_{i_1}(\mathbf{c}_0, B_0)$, we have $\mathbf{c}_k < 0$ for all $k = 1, \dots, n$.

In this paper, we study the maximal green sequences for the quivers which are mutation-equivalent to the quiver X_7 (Fig. 1). Our result is the following:

Theorem 1. Suppose that Q is mutation-equivalent to the quiver X_7 (so Q is one of the quivers in Fig. 1). Then Q does not have any maximal green sequences.

We prove the theorem using the following general statement, which can be easily checked to give a sufficient condition for the non-existence of maximal green sequences:

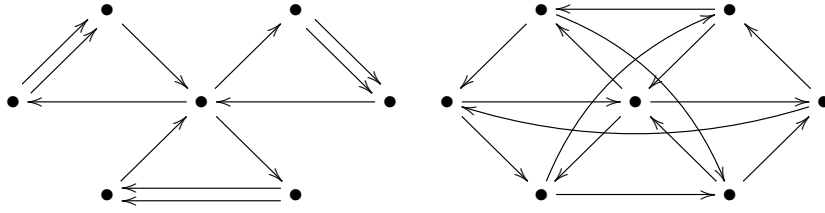


Figure 1. Quivers which are mutation-equivalent to X_7 ; the first one is the quiver X_7 , see [6].

Proposition 1. *Let B_0 be a skew-symmetrizable initial exchange matrix. Suppose that there is a vector $u > 0$ such that, for any Y -seed (\mathbf{c}, B) with respect to the initial seed (\mathbf{c}_0, B_0) , the coordinates of u with respect to \mathbf{c} are non-negative. Then, under assumption (3), the matrix B_0 does not have any maximal green sequences.*

We establish such a vector for the quiver X_7 :

Proposition 2. *Suppose that Q_0 is a quiver which is mutation-equivalent to X_7 , so Q_0 is one of the quivers in Fig. 1, and B_0 is the corresponding skew-symmetric matrix. Let $u = (a_1, a_2, \dots, a_7)$ be the vector defined as follows:*

(*) *if Q_0 is the quiver X_7 (so Q is the first quiver in Fig. 1), then the coordinate corresponding to the “center” is equal 2, and the rest is equal to 1; if Q_0 is not the quiver X_7 (so Q is the second quiver in Fig. 1), then all coordinates are equal to 1.*

Then, for any Y -seed (\mathbf{c}, B) with respect to the initial seed (\mathbf{c}_0, B_0) , the coordinates of u with respect to \mathbf{c} is of the same form as in (*). In particular, the coordinates of u with respect to \mathbf{c} are positive.

(The vector u is a radical vector for B_0 , i.e. $B_0 u = 0$. In fact, any radical vector for B_0 is a multiple of u .)

We generalize this statement to an arbitrary quiver as follows:

Theorem 2. *Let B_0 be a skew-symmetrizable initial exchange matrix and suppose that $u_0 > 0$ is a radical vector for B_0 , i.e. $B_0 u_0 = 0$. Suppose also that, for any Y -seed (\mathbf{c}, B) with respect to the initial seed (\mathbf{c}_0, B_0) , the coordinates of u_0 with respect to \mathbf{c} are non-negative. Then, under assumption (3), for any B which is mutation-equivalent to B_0 , the matrix B does not have any maximal green sequences.*

We prove our results in Section 2. For related applications of maximal green sequences, we refer the reader to [4] and [11].

2 Proofs of main results

Let us first note how the coordinates of a vector change under the mutation operation, which can be easily checked using the definitions (assuming (3)):

Proposition 3. *Suppose that (\mathbf{c}, B) is a Y -seed with respect to an initial Y -seed. Suppose also that the coordinate vector of u with respect to \mathbf{c} is (a_1, \dots, a_n) . Let $(\mathbf{c}', B') = \mu_k(\mathbf{c}, B)$ and (a'_1, \dots, a'_n) be the coordinates of u with respect to \mathbf{c}' . Then $a_i = a'_i$ if $i \neq k$ and $a'_k = -a_k + \sum a_i [\text{sgn}(\mathbf{c}_k) B_{k,i}]_+$, where the sum is over all $i \neq k$.*

As we mentioned, in view of Proposition 1, Theorem 1 follows from Proposition 2. To prove Proposition 2, it is enough to show that the coordinates of the vector u change as stated, i.e. show that if the coordinates of u with respect to \mathbf{c} are as in (*), then for the Y -seed $(\mathbf{c}', B') = \mu_k(\mathbf{c}, B)$, the coordinates with respect to \mathbf{c}' are also of the form in (*). This can be checked easily using the formula in Proposition 3.

To prove Theorem 2, let us first note the following property of the coordinates of the radical vectors:

Lemma 1. *Suppose that (\mathbf{c}, B) is a Y -seed with respect to an initial Y -seed (\mathbf{c}_0, B_0) and u_0 is a radical vector for B_0 . Suppose that the coordinate vector of u_0 with respect to \mathbf{c} is (a_1, \dots, a_n) . Then, for any index k , we have the following:*

$$\sum a_i [\text{sgn}(\mathbf{c}_k) B_{k,i}]_+ = \sum a_i [-\text{sgn}(\mathbf{c}_k) B_{k,i}]_+,$$

where the sum is over all $i \neq k$.

In particular, for radical vectors, the formula in Proposition 3 that describe the change of coordinates under mutation depends only on the exchange matrix, not on the \mathbf{c} -vectors.

To prove the lemma, suppose that $D = \text{diag}(d_1, \dots, d_n)$ is a skew-symmetrizing matrix for B_0 , so it is also skew-symmetrizing for B , so $DB = C$ is skew-symmetric, i.e. $C_{i,k} = d_i B_{i,k} = -d_k B_{k,i} = -C_{k,i}$ for all i, k . Let $u = (a_1, \dots, a_n)$. Then u is a radical vector for B , so it is also a radical vector for $C = DB$, i.e. $Cu = 0$, which means that for any index k , we have $\sum a_i [\text{sgn}(\mathbf{c}_k) C_{k,i}]_+ = \sum a_i [\text{sgn}(\mathbf{c}_k) C_{i,k}]_+$, which is equal to $\sum a_i [-\text{sgn}(\mathbf{c}_k) C_{k,i}]_+$, where the sum is over all $i \neq k$. Then, writing $C_{k,i} = d_k B_{k,i}$, we have

$$\sum a_i [\text{sgn}(\mathbf{c}_k) d_k B_{k,i}]_+ = \sum a_i [-\text{sgn}(\mathbf{c}_k) d_k B_{k,i}]_+.$$

Dividing both sides by $d_k > 0$, we obtain the lemma.

We will also need the following property of the radical vectors:

Lemma 2. *In the set-up of Theorem 2, let u denote the vector which represents u_0 with respect to the basis \mathbf{c} . Then u is a radical vector for B , i.e. $Bu = 0$.*

To prove the lemma, let us note that u can be obtained from u_0 by applying the formula in Proposition 3 along with the mutations. Thus, to prove the lemma, it is enough to show that, for any $k = 1, \dots, n$, we have the following:

(**) if $u = (a_1, \dots, a_n)$ is a radical vector for B , then u' is a radical vector for $B' = \mu_k(B)$, i.e. $B'u' = 0$, where $u' = (a'_1, \dots, a'_n)$ is the vector as in Proposition 3.

To show (**), we write B' in matrix notation as follows [2, Lemma 3.2]: for $\epsilon = \text{sgn}(\mathbf{c}_k)$, we have

$$B' = (J_{n,k} + E_k)B(J_{n,k} + F_k),$$

where

- $J_{n,k}$ denotes the diagonal $n \times n$ matrix whose diagonal entries are all 1's, except for -1 in the k th position;
- E_k is the $n \times n$ matrix whose only nonzero entries are $e_{ik} = [-\epsilon b_{ik}]_+$;
- F_k is the $n \times n$ matrix whose only nonzero entries are $f_{kj} = [\epsilon b_{kj}]_+$.

It follows from a direct check that $(J_{n,k} + F_k)u' = u$. Then $B'u' = (J_{n,k} + E_k)B(J_{n,k} + F_k)u' = (J_{n,k} + E_k)Bu = (J_{n,k} + E_k)0 = 0$. This completes the proof of the lemma.

Let us now prove Theorem 2. For this, we first consider the Y -seed pattern defined by the initial Y -seed (\mathbf{c}_0, B_0) at the initial vertex t_0 . Let us suppose that t_1 is a vertex such that the corresponding Y seed (\mathbf{c}, B) has the exchange matrix B . Then we can consider the Y -seed pattern defined by the initial Y -seed (\mathbf{c}_0, B) at the initial vertex t_1 (where \mathbf{c}_0 is the tuple of standard basis). Then we have the following: for any fixed vertex t of the n -regular tree \mathbb{T}_n , the exchange matrices of the Y -seeds assigned by these patterns coincide because the pattern is formed by mutating at the labels of the n -regular tree \mathbb{T}_n and mutation is an involutive operation on matrices; let us denote these seeds by (\mathbf{c}', B') and (\mathbf{c}'', B') respectively.

On the other hand, let u denote the vector which represents u_0 with respect to the basis \mathbf{c} , which can be obtained by applying the formula in Proposition 3 along with the mutations. Then u is a radical vector for B , i.e. $Bu = 0$ (Lemma 2). Furthermore, the coordinates of the vectors u_0 and u with respect to the bases \mathbf{c}' and \mathbf{c}'' respectively will coincide by Lemma 1 (which says that for radical vectors the formula in Proposition 3 depends only on the exchange matrices, not on the \mathbf{c} -vectors). In particular, the coordinates of u with respect to any basis of \mathbf{c} -vectors are non-negative. Thus, by Proposition 1, the matrix B does not have any maximal green sequences. This completes the proof.

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